Bound solitons in the ac-driven, damped nonlinear Schrödinger equation

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We demonstrate analytically that the effective potential of interaction between widely separated solitons in a damped, ac-driven nonlinear Schrödinger equation is oscillatory at large distances. We show numerically that two solitons in the system attract each other if the initial distance between them is smaller than a certain critical value; consequently, they will either form an oscillatory bound state with a finite lifetime, collapse to a stable single soliton state, or decay to the rotating background. If the initial separation is greater than the critical value, they separate to form a stable bound state at a second critical distance. The critical distances are in good agreement with the analytical prediction.

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The interaction of solitons with slightly overlapping tails in the unperturbed (1+1)-dimensional nonlinear Schrödinger (NLS) system is well known [1]. In the unperturbed system the interaction potential of solitons does not have a local minimum. However, recent analytical [4–6] and numerical work [7] has shown possible existence of bound solitons in various perturbed NLS and related systems. This arises from the fact that the interaction potential is oscillatory at large distances and therefore capable of binding solitons to form bound states.

Here, we study the bound soliton state in the perturbed NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = -i\alpha u + \epsilon e^{i\omega t}, \tag{1}$$

where the driving frequency ω and the dissipation constant α are positive. Equation (1) is one of the simplest dynamical models for a number of nonlinear physical systems [2]. It has been demonstrated by means of perturbation theory [3] that Eq. (1) supports a soliton with internal frequency ω phase locked to the ac-driving term:

$$u_{\rm sol} = \sqrt{|\omega|}\,\mathrm{sech}(\sqrt{|\omega|}x)\,\exp(i\omega t + i\delta), \tag{2}$$

where

$$\sin\delta = \frac{2\alpha\sqrt{|\omega|}}{\pi\epsilon}. \tag{3}$$

The soliton exists provided $|\sin\delta| < 1$, i.e., if the drive amplitude exceeds the threshold value

$$\epsilon_{\rm thr} = 2\pi^{-1}\alpha\sqrt{|\omega|}. \tag{4}$$

This approximation is valid provided $\alpha \ll |\omega|$ and $\epsilon \ll |\omega|^{3/2}$. Beyond the limits of applicability of the perturbation theory, one cannot find the form of a phase-locked soliton explicitly. However, it is possible to prove that it exists. This is a soliton riding on an oscillating background, $u_b(t) = u_0 e^{i\omega t}$, in which u_0 satisfies

$$-\omega u_0 + 2u_0|u_0|^2 = -i\alpha u_0 + \epsilon. \tag{5}$$

In the following, we analyze the interaction between two solitons in the damped ac-driven NLS (1) [6]. To obtain the asymptotic form of the soliton solution in this model far from the center of the soliton, we can linearize Eq. (1) to derive the following expression:

$$u = u_{\rm sol} + u_b\,,$$
$$u_{\rm sol} \approx 2\eta\exp(-\eta|x-z_n| + i\eta^2 t - ik|x-z_n| + \phi_n), \tag{6}$$

where z_n is the coordinate of the center of the nth soliton, $\eta = \sqrt{|\omega|}$ is its amplitude, and ϕ_n is the phase which locks to the external driving according to Eq. (3). The wave number k satisfies

$$k = -\mathrm{Im}[\sqrt{\eta^2 - i\alpha}] \approx \frac{\alpha}{2\eta}, \tag{7}$$

which is solely produced by the small perturbation terms in Eq. (1). Next, considering interaction between widely separated solitons, we insert into the interaction Hamiltonian

$$H_{\rm int} = -\int_{-\infty}^{+\infty} |u(x)|^4 dx \tag{8}$$

the approximate field $u(x,t) = u_{1,\rm sol}(x,t) + u_{2,\rm sol}(x,t)$ to obtain the effective soliton-soliton interaction potential

$$U = -4\int_{-\infty}^{+\infty} |u_{1,\rm sol}(x)|^2 \mathrm{Re}[u_{1,\rm sol}(x)u^*_{2,\rm sol}(x)]dx + (1 \leftrightarrow 2). \tag{9}$$

Here $u_{1,\rm sol}$ is the locked soliton [see Eq. (2)] and $u_{2,\rm sol}$ is the the tail of the other locked soliton as expressed in Eq. (6). From this, we have

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$$U = -16\eta^4 \cos\phi \int_{-\infty}^{+\infty} dx\, \mathrm{sech}^3\eta x\, e^{-\eta|x-z|} \cos(k|x-z|)$$
$$\approx -32\eta^3 \exp(-\eta z) \cos(kz) \cos\phi, \tag{10}$$

where $z \equiv z_1 - z_2$ and $\phi \equiv \phi_1 - \phi_2$ is the phase difference between the solitons, which vanishes since the solitons are all phase locked to the external ac driver. In the last step in Eq. (10) we have assumed that $\eta z \gg 1$ and $k/\eta \ll 1$. For $k = 0$, Eq. (10) reduces to the interaction potential between two solitons for the unperturbed NLS.

It is straightforward to see (for $k \ll \eta$) that the potential (10) has local extrema at the points

$$z_n \approx \frac{\pi}{2k}(1 + 2n), \tag{11}$$

where n is a nonnegative integer. The first extremum, $n = 0$, is a maximum,

$$z_{\max} = \frac{\pi}{2k}, \tag{12}$$

which is much larger than the characteristic width of the soliton $\sim \eta^{-1}$. It is well known that identical solitons with zero phase difference attract each other in the unperturbed NLS equation, executing a periodic oscillation in their separation [1]. At large separations, this unperturbed interaction is altered by the perturbation terms and the potential becomes oscillatory. Thus we can predict a critical separation between the two solitons given by Eq. (12) such that the solitons attract each other if $z < z_{\max}$ and they effectively repel each other in the opposite case. In the latter case, they gradually move to the second extremum, which is a minimum,

$$z_{\min} = \frac{3\pi}{2k}, \tag{13}$$

where they form a bound soliton state. When the damping is sufficiently weak, it can be expected that the two attracting solitons will form an oscillatory bound state in the first well of the potential. However, they can only live for a finite time since each time they collide with one another they will deform and are no longer able to maintain the locked soliton solution [see Eq. (2)]. Therefore they will gradually lose energy and decay. If the damping is not sufficiently weak, the two solitons may collide and radiate enough energy to form a stable soliton. If the damping is further increased, even this single soliton state will no longer be able to sustain itself and the two solitons will decay to the rotating background. Furthermore, by noting that the binding energy E of the bound state is exponentially small,

$$E_n = -U(z = z_n) \approx 32k\eta^2 \exp\left(-\frac{(1+2n)\eta\pi}{2k}\right), \tag{14}$$

where n is an odd integer, we can expect that it becomes harder and harder to detect effects of the alternating barriers and wells in the potential with increasing n.

To verify these predictions, we have performed numerical simulations of the soliton-soliton interactions in the model. The simulations have confirmed the above predictions.

(a) In Fig. 1, we show evolution of the separation between two solitons in time. The curves are the temporal traces of the position of the density maximum of the solitons. When the initial separation is chosen smaller than the critical one (the curve with intermediate thickness in Fig. 1), the solitons move towards each other, and collapse into a new single soliton. If the initial separation is chosen slightly larger than the critical distance (thickest curve), they slowly separate. For comparison, we have plotted the periodic trajectory of the bound state of two unperturbed solitons (the finest curve). The missing part of the curve is due to the fact that our tracing window was not fine enough to record the fast collapse of two maxima into the collision. Obviously, when they pass through each other, the temporal evolution is not singly peaked: for clarity the figure omits the traces of the smaller peaks. In Fig. 2, we show a bound state in the first well of the effective potential. It clearly decays in time. We observed that the amplitude of the two solitons gradually decreases. The final jiggling part of the trace is when the two solitons gradually merge with the rotating background. We have also observed that the two

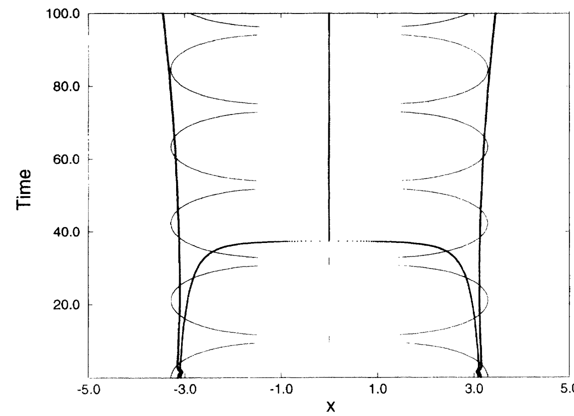

FIG. 1. Time dependence of the separation between two solitons for different initial distances with $\alpha = 0.50$, $\omega = 1.00$ (see text).

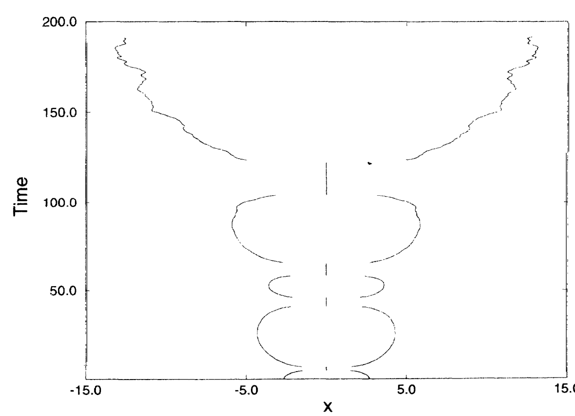

FIG. 2. Example of a bound state with a finite lifetime in the first well of the effective potential, $\alpha = 0.005$, $\omega = 1.00$. Shown here is the temporal trace of the position of the maximum of the solitons.

solitons decay into the rotating background after the first collision if the damping is strong enough. As an aside, we comment on the ansatz (6) of solitons riding on a rotating background. For the repelling soliton case, we have measured the rotating background amplitudes far away from the solitons and compared these with the solutions of Eq. (5). They are well within 0.5% relative error. This further justifies Eq. (6).

(b) To estimate the accuracy of the analytical expression (12), we display in Fig. 3 the dependence of the critical distance $z_{\max}$ on the dissipation constant α. As can be seen from this figure, the analytical prediction is in good agreement with the numerical results, especially considering that we used relatively large dissipation parameters for numerical convenience. In the measurement, we fixed the driving frequency $\omega = 1$ for all dissipation parameters. Notice that, due to the scaling property of NLS, we can always rescale ω to one and leave only the parameters α and ϵ free in Eq. (1). Taking into account the fact that the theoretical threshold (4) tends to be lower than the one observed we chose the strength ϵ to be slightly higher than the theoretical threshold with the relative difference between them less than 1%. In the above derivation of the effective potential, the driving strength ϵ does not have any significance as long as it is sufficiently strong to sustain a soliton against damping. As is well known, there is a range of driving strengths which can lock solitons in the driven, damped system [8]. In the numerical simulations, we have tested the effects of increasing ϵ on the interaction of solitons. For example, in the case of $\alpha = 0.40$, $\omega = 1.00$ the locking range is $0.257 < \epsilon < 0.33$ above which multiple solitons can be excited. By increasing ϵ from 0.257 we observed that during the initial transient period the two solitons acquire a larger and larger oscillatory displacement with respect to their respective initial positions, and the oscillation frequency is identical to that of the external driver. A strong oscillatory transient motion will sufficiently deform the solitons away from our ansatz (6) and causes further complications in the soliton-soliton interaction. Consequently, to achieve a good comparison between theory and simulation, care should be taken in choosing ϵ to minimize the additional effects in the soliton-soliton interaction. Finally, we note that the overall agreement between the numerical results and theory would be of 15%, if the evaluation in Eqs. (10) and (11) were done numerically without the approximations we used in order to obtain these explicit equations.

(c) We have detected the stable bound state trapped in the second well whose separation is 13.0 for $\alpha = 0.6$ and $\omega = 1$. Comparing with $z_{\min} = 15.7$ as predicted, there is a relative error of 18%. In this case the damping α is relative large, again the deviation from the analytical prediction is expected. The potential at this distance is five orders of magnitude smaller than at $z_{\max}$ and is extremely weak compared to the soliton energy. In numerical simulations, one might be concerned with possible discrete lattice pinning effects. We have checked that the bound state with exponentially small binding energy is not an artifact of discreteness. By shifting the initial configuration a half lattice spacing the results for the bound state were all exactly reproduced.

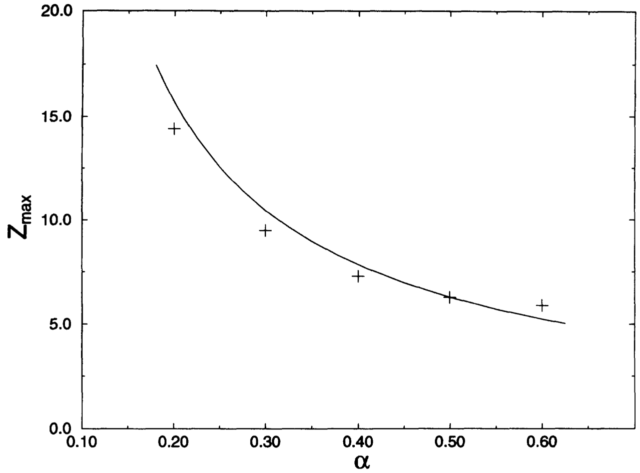

FIG. 3. Critical separation between two solitons vs α with $\omega = 1.00$. +, results of numerical simulation; solid line, analytical predication.

In conclusion, we have shown that the effective potential of interaction of two widely separated solitons in the damped, ac-driven (1+1)-dimensional NLS system is oscillatory at large distances. We have presented the numerical results on bound states and compared them with predictions of an effective potential theory. As a quantitative test, the critical distances for the formation of bound states obtained in numerical simulation are shown to be in good agreement with the analytical predictions.

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